

ON THE BOUNDARY BEHAVIOR OF THE CURVATURE OF L^2 -METRICS

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ABSTRACT. For one-parameter degenerations of compact Kähler manifolds, we determine the asymptotic behavior of the first Chern form of the direct image of a Nakano semi-positive vector bundle twisted by the relative canonical bundle, when the direct image is equipped with the L^2 -metric.

1. INTRODUCTION

Let X be a connected Kähler manifold of dimension $n+1$ with Kähler metric h_X and let $S = \{s \in \mathbf{C}; |s| < 1\}$ be the unit disc. Set $S^\circ := S \setminus \{0\}$. Let $\pi: X \rightarrow S$ be a proper surjective holomorphic map with connected fibers. Let Σ_π be the critical locus of π . We assume that $\pi(\Sigma_\pi) = \{0\}$. We set $X_s = \pi^{-1}(s)$ for $s \in S$. Then X_s is non-singular for $s \in S^\circ$. Let $\omega_X = \Omega_X^{n+1}$ be the canonical bundle of X and let $\omega_{X/S} = \Omega_X^{n+1} \otimes (\pi^* \Omega_S^1)^{-1}$ be the relative canonical bundle of $\pi: X \rightarrow S$. The Kähler metric h_X induces a Hermitian metric $h_{X/S}$ on $TX/S = \ker \pi_*|_{X \setminus \Sigma_\pi}$, and $h_{X/S}$ induces a Hermitian metric $h_{\omega_{X/S}}$ on $\omega_{X/S}$.

Let $\xi \rightarrow X$ be a holomorphic vector bundle on X equipped with a Hermitian metric h_ξ . We write $\omega_{X/S}(\xi) = \omega_{X/S} \otimes \xi$. In this note, we assume that (ξ, h_ξ) is a Nakano semi-positive vector bundle on X . Namely, if R^ξ denotes the curvature form of (ξ, h_ξ) with respect to the holomorphic Hermitian connection, then the Hermitian form $h_\xi(\sqrt{-1}R^\xi(\cdot), \cdot)$ on the holomorphic vector bundle $TX \otimes \xi$ is semi-positive. Since $\dim S = 1$ and since (ξ, h_ξ) is Nakano semi-positive, all direct image sheaves $R^q \pi_* \omega_{X/S}(\xi)$ are locally free by [12]. By the fiberwise Hodge theory, $R^q \pi_* \omega_{X/S}(\xi)$ is equipped with the L^2 -metric h_{L^2} with respect to $h_{X/S}$ and $h_{\omega_{X/S}} \otimes h_\xi$. By Berndtsson [2] and Mourougane-Takayama [7], the holomorphic Hermitian vector bundle $(R^q \pi_* \omega_{X/S}(\xi), h_{L^2})$ is again Nakano semi-positive on S° . By Mourougane-Takayama [8], h_{L^2} induces a singular Hermitian metric with semi-positive curvature current on the tautological quotient bundle over the projective-space bundle $\mathbf{P}(R^q \pi_* \omega_{X/S}(\xi))$. (We remark that there is no restrictions of the dimension of the base space S in the works [2], [7], [8].)

After these results, one of the natural problems to be considered is the quantitative estimates for the singularities of the L^2 -metric and its curvature. In [14], we gave a formula for the singularity of the L^2 -metric on $R^q \pi_* \omega_{X/S}(\xi)$ (cf. Sect. 2). As a consequence, if σ_q is a nowhere vanishing holomorphic section of $\det R^q \pi_* \omega_{X/S}(\xi)$, then there exist a rational number $a_q \in \mathbf{Q}$, an integer $\ell_q \geq 0$ and a real number c_q such that (cf. [14, Th. 6.8])

$$\log \|\sigma_q(s)\|_{L^2}^2 = a_q \log |s|^2 + \ell_q \log(-\log |s|^2) + c_q + O(1/\log |s|) \quad (s \rightarrow 0).$$

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In this note, we study the boundary behavior of the curvature of the holomorphic Hermitian vector bundle $(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$ as an application of the description of the singularity of the L^2 -metric h_{L^2} given in [14]. In this sense, this note is a supplement to the article [14].

Let us state our results. Let $\mathcal{R}(s) ds \wedge d\bar{s}$ be the curvature form of $R^q\pi_*\omega_{X/S}(\xi)$ with respect to the holomorphic Hermitian connection associated to h_{L^2} . By the Nakano semi-positivity [2], [7], $\sqrt{-1}\mathcal{R}(s)$ is a semi-positive Hermitian endomorphism on the Hermitian bundle $(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$ on S° .

Theorem 1.1. *The curvature form $\mathcal{R}(s) ds \wedge d\bar{s}$ has Poincaré growth near $0 \in S$. Namely, there exists a constant $C > 0$ such that the following inequality of Hermitian endomorphisms holds for all $s \in S^\circ$*

$$0 \leq \sqrt{-1}\mathcal{R}(s) \leq \frac{C}{|s|^2(\log|s|)^2} \text{Id}_{R^q\pi_*\omega_{X/S}(\xi)}.$$

Moreover, the Chern form $c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$ has the following asymptotic behavior as $s \rightarrow 0$:

$$c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2}) = \left\{ \frac{\ell_q}{|s|^2(\log|s|)^2} + O\left(\frac{1}{|s|^2(\log|s|)^3}\right) \right\} \sqrt{-1} ds \wedge d\bar{s}.$$

Considering the trivial example $X = M \times S$, $\xi = \mathcal{O}_X$, $\pi = \text{pr}_2$, where M is a compact Kähler manifold, we can not expect any lower bound of $\sqrt{-1}\mathcal{R}(s)$ (resp. $c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$) by a non-zero semi-positive Hermitian endomorphism (resp. real $(1,1)$ -form). We remark that, when X_0 is reduced and has only canonical singularities, then we get a better estimate (cf. Sect. 5).

As an application of Theorem 1.1, we get an estimate for the complex Hessian of analytic torsion. Set $X_s := \pi^{-1}(s)$ and $\xi_s := \xi|_{X_s}$ for $s \in S$. Let ω_{X_s} be the canonical line bundle of X_s and let $h_{\omega_{X_s}}$ be the Hermitian metric on ω_{X_s} induced from h_X . For $s \in S^\circ$, let $\tau(X_s, \omega_{X_s}(\xi_s))$ be the analytic torsion [10], [3] of the holomorphic Hermitian vector bundle $(\xi_s \otimes \omega_{X_s}, h_{\xi|_{X_s}} \otimes h_{\omega_{X_s}})$ on the compact Kähler manifold $(X_s, h_X|_{X_s})$. Let $\log \tau(X/S, \omega_{X/S}(\xi))$ be the function defined as

$$\log \tau(X/S, \omega_{X/S}(\xi))(s) := \log \tau(X_s, \omega_{X_s}(\xi_s)), \quad s \in S^\circ.$$

By Bismut-Gillet-Soulé [3], $\log \tau(X/S, \omega_{X/S}(\xi))$ is a C^∞ function on S° . Moreover, under certain algebraicity assumption of the family $\pi: X \rightarrow S$ and the vector bundle ξ , there exist by [14] constants $\alpha \in \mathbf{Q}$, $\beta \in \mathbf{Z}$, $\gamma \in \mathbf{R}$ such that

$$\log \tau(X/S, \omega_{X/S}(\xi))(s) = \alpha \log |s|^2 - \left(\sum_{q \geq 0} (-1)^q \ell_q \right) \log(-\log |s|^2) + \gamma + O(1/\log |s|)$$

as $s \rightarrow 0$. By this asymptotic expansion, it is reasonable to expect that the complex Hessian of analytic torsion has a similar behavior to the Poincaré metric on S° .

Theorem 1.2. *The complex Hessian $\partial_{s\bar{s}} \log \tau(X/S, \omega_{X/S}(\xi))$ has the following asymptotic behavior as $s \rightarrow 0$:*

$$\partial_{s\bar{s}} \log \tau(X/S, \omega_{X/S}(\xi)) = \frac{\sum_{q \geq 0} (-1)^q \ell_q}{|s|^2(\log|s|)^2} + O\left(\frac{1}{|s|^2(\log|s|)^3}\right).$$

This note is organized as follows. In Sect. 2, we recall the structure of the singularity of the L^2 -metric h_{L^2} on $R^q\pi_*\omega_{X/S}(\xi)$. In Sect. 3, we prove some technical lemmas used in the proof of Theorem 1.1. In Sect. 4, we prove Theorems 1.1 and 1.2. In Sect. 5, we study the case where X_0 has only canonical singularities.

Throughout this note, we keep the notation and the assumptions in Sect. 1.

2. THE SINGULARITY OF THE L^2 -METRIC

2.1. The structure of the singularity of the L^2 -metric. Let $\kappa_{\mathcal{X}}$ be the Kähler form of $h_{\mathcal{X}}$. In the rest of this note, we assume that (ξ, h_{ξ}) is *Nakano semi-positive on X and that $(S, 0) \cong (\Delta, 0)$* . By [12, Th. 6.5 (i)], $R^q \pi_* \omega_{X/S}(\xi)$ is locally free on S . By shrinking S if necessary, we may also assume that $R^q \pi_* \omega_{X/S}(\xi)$ is a free \mathcal{O}_S -module on S . Let $\rho_q \in \mathbf{Z}_{\geq 0}$ be the rank of $R^q \pi_* \Omega_X^{n+1}(\xi)$ as a free \mathcal{O}_S -module on S . Let $\{\psi_1, \dots, \psi_{\rho_q}\} \subset H^0(S, R^q \pi_* \omega_{X/S}(\xi))$ be a free basis of the locally free sheaf $R^q \pi_* \omega_{X/S}(\xi)$ on S .

Let T be another unit disc. By the semistable reduction theorem [9, Chap II], there exists a positive integer $\nu > 0$ such that the family $X \times_S T \rightarrow T$ induced from $\pi: X \rightarrow S$ by the map $\mu: T \rightarrow S$, $\mu(t) = t^{\nu}$, admits a semistable model. Namely, there is a resolution $r: Y \rightarrow X \times_S T$ such that the family $f := \text{pr}_2 \circ r: Y \rightarrow T$ is semistable, i.e., $Y_0 := f^{-1}(0)$ is a *reduced* normal crossing divisor of Y . We fix such an integer $\nu > 0$. Let $\text{Herm}(r)$ be the set of $r \times r$ -Hermitian matrices.

Theorem 2.1. *By choosing a basis $\{\psi_1, \dots, \psi_{\rho_q}\}$ of $R^q \pi_* \omega_{X/S}(\xi)$ as a free \mathcal{O}_S -module appropriately, the $\rho_q \times \rho_q$ -Hermitian matrix*

$$G(s) := (h_{L^2}(\psi_{\alpha}|_{X_s}, \psi_{\beta}|_{X_s}))$$

has the following expression

$$G(t^{\nu}) = D(t) \cdot H(t) \cdot \overline{D(t)}, \quad D(t) = \text{diag}(t^{-e_1}, \dots, t^{-e_{\rho_q}}).$$

Here $e_1, \dots, e_{\rho_q} \geq 0$ are integers and the Hermitian matrix $H(t)$ has the following structure: There exist $A_m(t) \in C^{\infty}(T, \text{Herm}(\rho_q))$, $0 \leq m \leq n$, with

$$H(t) = \sum_{m=0}^n A_m(t) (\log |t|^2)^m.$$

Moreover, by defining the real-valued functions $a_m(t) \in C^{\infty}(T)$, $0 \leq m \leq n\rho_q$ as

$$\det H(t) = \sum_{m=0}^{n\rho_q} a_m(t) (\log |t|^2)^m,$$

one has $a_m(0) \neq 0$ for some $0 \leq m \leq n\rho_q$.

Proof. See [14, Th. 6.8 and Lemmas 6.3 and 6.4]. □

Remark 2.2. The meaning of the Hermitian matrix $H(t)$ and the diagonal matrix $D(t)$ is explained as follows. Let $F: Y \rightarrow X$ be the map defined as the composition of $r: Y \rightarrow X \times_S T$ and $\text{pr}_1: X \times_S T \rightarrow X$. Then $h_Y := r^*(h_X + dt \otimes d\bar{t})$ is a Kähler metric on $Y \setminus Y_0$. There is a basis $\{\theta_1, \dots, \theta_{\rho_q}\}$ of $R^q f_* \omega_{Y/Y}(F^* \xi)$ such that $H(t) = (H_{\alpha\bar{\beta}}(t))$, $H_{\alpha\bar{\beta}}(t) = (\mu^* h_{L^2})(\theta_{\alpha}, \theta_{\beta})$, where $\mu^* h_{L^2}$ is the L^2 -metric on $R^q f_* \omega_{Y/T}(F^* \xi)$ with respect to h_Y and $F^* h_{\xi}$. By [8, Lemma 3.3], $R^q f_* \omega_{Y/T}(F^* \xi)$ is regarded as a subsheaf of $\mu^* R^q f_* \omega_{X/S}(\xi)$. Then the relation between the two basis $\{\theta_1, \dots, \theta_{\rho_q}\}$ and $\{\mu^* \psi_1, \dots, \mu^* \psi_{\rho_q}\}$ is given by $D(t)$, i.e., $\theta_{\alpha} = t^{e_{\alpha}} \mu^* \psi_{\alpha}$. Moreover, by [8, Lemma 4.2], $\mu^* h_{L^2}|_{T^{\circ}}$ is indeed the pull-back of the L^2 -metric $h_{L^2}|_{S^{\circ}}$ via μ , where $T^{\circ} := T \setminus \{0\}$, which implies the relation $G(\mu(t)) = D(t)H(t)\overline{D(t)}$.

The proof of Theorem 2.1 heavily relies on a theorem of Barlet [1, Th. 4 bis.]. This is the major reason why we need the assumption $\dim S = 1$.

2.2. A Hodge theoretic proof of Theorem 2.1 for a trivial line bundle.

Assume that (ξ, h_ξ) is a trivial Hermitian line bundle on X , that $\pi: X \rightarrow S$ is a family of polarized projective manifolds with unipotent monodromy and that the Kähler class of h_X is the first Chern class of an ample line bundle on X . We see that the expansion in Theorem 2.1 follows from the nilpotent orbit theorem of Schmid [11] in this case.

Let κ_X be the Kähler class of h_X . By assumption, there is a very ample line bundle L on X with $[\kappa_X] = c_1(L)/N$. Replacing κ_X by $N\kappa_X$ if necessary, we may assume that L is very ample. Let $H_1, \dots, H_n \in |L|$ be sufficiently generic hyperplane sections such that the following hold for all $0 \leq k \leq n$ after shrinking S if necessary:

- (i) $X \cap H_1 \cap \dots \cap H_k$ is a complex manifold of dimension $n - k + 1$.
- (ii) The restriction of π to $X \cap H_1 \cap \dots \cap H_k$ is a flat holomorphic map from $X \cap H_1 \cap \dots \cap H_k$ to S .
- (iii) $X_s \cap H_1 \cap \dots \cap H_k$ is a projective manifold of dimension $n - k$ for $s \in S^\circ$.

We set $X_s^{(k)} := (\pi^{(k)})^{-1}(s) = X_s \cap H_1 \cap \dots \cap H_k$ for $s \in S$.

Let $\{\psi_1, \dots, \psi_{\rho_q}\} \subset H^0(S, R^q \pi_* \omega_{X/S})$ be a free basis of the locally free sheaf $R^q \pi_* \omega_{X/S}$ on S . There exists $\Psi_1, \dots, \Psi_{\rho_q} \in H^0(X, \Omega_X^{n+1-q})$ by [12, Th. 5.2] (after shrinking S if necessary) such that

$$\psi_\alpha = [(\Psi_\alpha \wedge \kappa_X^q) \otimes (\pi^* ds)^{-1}], \quad \pi^* ds \wedge \Psi_\alpha = 0.$$

By the condition $\pi^* ds \wedge \Psi_\alpha = 0$, there exist relative holomorphic differentials $\psi'_\alpha \in H^0(X \setminus \Sigma_\pi, \Omega_{X/S}^{n-q})$ such that $\Psi_\alpha = \psi'_\alpha \wedge \pi^* ds$. Then the harmonic representative of the cohomology class $\psi|_{X_s}$ is given by $\psi'_\alpha \wedge \kappa_X|_{X_s}$. Since $\kappa_X = c_1(L)$, we get

$$h_{L^2}(\psi_\alpha, \psi_\beta)(s) = i^{(n-q)^2} \int_{X_s} \psi'_\alpha \wedge \overline{\psi'_\beta} \wedge \kappa_X^q|_{X_s} = i^{(n-q)^2} \int_{X_s^{(q)}} \psi'_\alpha \wedge \overline{\psi'_\beta}|_{X_s^{(q)}}.$$

Hence Theorem 2.1 is reduced to the case $q = 0$. In the case $q = 0$, Theorem 2.1 is a consequence of Fujita's estimate [4, 1.12] and the following:

Lemma 2.3. *For $\varphi, \psi \in H^0(X, \Omega_X^{n+1})$, there exist $a_m(s) \in C^\omega(S)$, $0 \leq m \leq n$ such that*

$$\pi_*(\varphi \wedge \overline{\psi})(s) = \sum_{m=0}^n (\log |s|^2)^m a_m(s) ds \wedge d\bar{s}.$$

In particular, $h_{L^2}(\varphi \otimes (\pi^ ds)^{-1}|_{X_s}, \psi \otimes (\pi^* ds)^{-1}|_{X_s}) = i^{n^2} \sum_{m=0}^n (\log |s|^2)^m a_m(s)$.*

Proof. Fix $\mathfrak{o} \in S^\circ$. Let $\gamma \in GL(H^n(X_\mathfrak{o}, \mathbf{C}))$ be the monodromy. By assumption, γ is unipotent. Set $\mathbf{H} := R^n \pi_* \mathbf{C} \otimes_{\mathbf{C}} \mathcal{O}_{S^\circ}$, which is equipped with the Gauss-Manin connection. Let $\{v_1, \dots, v_m\}$ be a basis of $H^n(X_\mathfrak{o}, \mathbf{C})$. Since γ is unipotent, there exists a nilpotent $N \in \text{End}(H^n(X_\mathfrak{o}, \mathbf{C}))$ such that $\gamma = \exp(N)$. Let $p: \widetilde{S^\circ} \ni z \rightarrow \exp(2\pi iz) \in S^\circ$ be the universal covering. Since \mathbf{H} is flat, v_k extend to flat sections $\mathbf{v}_k \in \Gamma(\widetilde{S^\circ}, p^* \mathbf{H})$, which induces an isomorphism $p^* \mathbf{H} \cong \mathcal{O}_{\widetilde{S^\circ}} \otimes_{\mathbf{C}} H^n(X_\mathfrak{o}, \mathbf{C})$. Under this trivialization, we have $\mathbf{v}_k(z+1) = \gamma \cdot \mathbf{v}_k(z)$. We define $\mathbf{s}_k(\exp 2\pi iz) := \exp(-zN) \mathbf{v}_k(z)$. Then $\mathbf{s}_1, \dots, \mathbf{s}_m \in \Gamma(\widetilde{S^\circ}, p^* \mathbf{H})$ descend to single-valued holomorphic frame fields of \mathbf{H} . The canonical extension of \mathbf{H} is the locally free sheaf on S defined as $\overline{\mathbf{H}} := \mathcal{O}_S \mathbf{s}_1 \oplus \dots \oplus \mathcal{O}_S \mathbf{s}_m$. Set $\mathbf{F}^n := \pi_* \Omega_{X/S}^n|_{S^\circ} \subset \mathbf{H}$. By [11, p. 235], \mathbf{F}^n extends to a subbundle $\overline{\mathbf{F}}^n \subset \overline{\mathbf{H}}$.

There exists $\varphi', \psi' \in H^0(X \setminus X_0, \Omega_{X/S}^n|_{X \setminus X_0})$ such that $\varphi = \pi^* ds \wedge \varphi'$ and $\psi = \pi^* ds \wedge \psi'$ on $X \setminus X_0$. Then φ' and ψ' are identified with $\varphi \otimes (\pi^* ds)^{-1}, \psi \otimes (\pi^* ds)^{-1} \in H^0(X, \omega_{X/S})$, respectively. Since $\mathbf{F}^n \subset \mathbf{H}$, there exist $b_k(t), c_k(t) \in \mathcal{O}(S^\circ)$ such that $[\varphi'|_{X_s}] = \sum_{k=1}^m b_k(s) \mathbf{s}_k(s)$ and $[\psi'|_{X_s}] = \sum_{k=1}^m c_k(s) \mathbf{s}_k(s)$. Since $\pi_* \omega_{X/S} = \overline{F}^n$ by Kawamata [5, Lemma 1], we get $b_k(s), c_k(s) \in \mathcal{O}(S)$. Then

$$\pi_*(\varphi \wedge \overline{\psi})(s) = \left\{ \int_{X_s} \varphi' \wedge \overline{\psi'} \right\} ds \wedge d\bar{s} = \left\{ \int_{X_s} \sum_{j=1}^m b_j(s) \mathbf{s}_j(s) \wedge \sum_{k=1}^m \overline{c_k(s) \mathbf{s}_k(s)} \right\} ds \wedge d\bar{s}.$$

Substituting $\mathbf{s}_k(s) = \exp(-zN) \mathbf{v}_k(z) = \sum_{0 \leq m \leq n} \frac{(-z)^m}{m!} N^m \mathbf{v}_k(z)$, we get

$$\pi_*(\varphi \wedge \overline{\psi})(s) = \left\{ \sum_{j,k=1}^m b_j(s) \overline{c_k(s)} \sum_{0 \leq a,b \leq n} \frac{(-1)^{a+b}}{a!b!} z^a \bar{z}^b C_{a,b}^{j,k} \right\} ds \wedge d\bar{s},$$

where $z = \frac{1}{2\pi i} \log s$ and $C_{a,b}^{j,k} = \int_{X_s} (N^a v_j) \wedge (N^b \overline{v_k})$. Since $\pi_*(\varphi \wedge \overline{\psi})$ is single-valued, so is the expression $\sum_{a+b=m} \frac{(-1)^{a+b}}{a!b!} z^a \bar{z}^b C_{a,b}^{j,k}$. As a result, there exists a constant $C_m^{j,k} \in \mathbf{C}$ such that $\sum_{a+b=m} \frac{(-1)^{a+b}}{a!b!} z^a \bar{z}^b C_{a,b}^{j,k} = C_m^{j,k} (\log |s|^2)^m$. Setting $a_m(s) := \sum_{j,k=1}^m C_m^{j,k} b_j(s) \overline{c_k(s)}$, we get the result. \square

Remark 2.4. In the proof of Theorem 2.1, the role of the nilpotent orbit theorem is played by Barlet's theorem [1, Th. 4bis.] on the asymptotic expansion of fiber integrals associated to the function $f(z) = z_0 \cdots z_n$ near the origin. See [14, Sects. 6.3 and 6.4] for more details.

3. SOME TECHNICAL LEMMAS

We denote by $C_{\mathbf{R}}^\infty(T)$ the set of real-valued C^∞ functions on T .

Lemma 3.1. *Let $\varphi(t) \in C_{\mathbf{R}}^\infty(T)$ and let $r \in \mathbf{Q}$ and $\ell \in \mathbf{Z}$. Set $h(t) := |t|^{2r} (\log |t|^2)^\ell \varphi(t)$. Then the following identities hold:*

$$\begin{aligned} (1) \quad \partial_t h(t) &= \left(\frac{r}{t} + \frac{\ell}{t(\log |t|^2)} + \frac{\partial_t \varphi(t)}{\varphi(t)} \right) h(t), \\ (2) \quad \partial_{t\bar{t}} h(t) &= \left(-\frac{\ell}{|t|^2 (\log |t|^2)} + \frac{\partial_{t\bar{t}} \varphi(t)}{\varphi(t)} - \frac{|\partial_t \varphi(t)|^2}{\varphi(t)^2} + \left| \frac{r}{t} + \frac{\ell}{t(\log |t|^2)} + \frac{\partial_t \varphi(t)}{\varphi(t)} \right|^2 \right) h(t). \end{aligned}$$

Proof. The proof is elementary and is left to the reader. \square

Lemma 3.2. *Let I be a finite set. For $i \in I$, let $r_i \in \mathbf{Q}$, $\ell_i \in \mathbf{Z}$ and $\varphi_i(t) \in C_{\mathbf{R}}^\infty(T)$. Set $g_i(t) := |t|^{2r_i} (\log |t|^2)^{\ell_i} \varphi_i(t)$ for $i \in I$ and $g(t) := \sum_{i \in I} g_i(t)$.*

(1) *If $g(t) > 0$ on T° , then the following equalities of functions on T° hold:*

$$\begin{aligned} \partial_t \log g &= \sum_{i \in I} \left(\frac{r_i}{t} + \frac{\ell_i}{t(\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right) \frac{g_i}{g}, \\ \partial_{t\bar{t}} \log g &= -\frac{1}{2} \sum_{i,j} \frac{\ell_i + \ell_j}{|t|^2 (\log |t|^2)^2} \cdot \frac{g_i g_j}{g^2} + \frac{1}{2} \sum_{i,j} \left| \frac{r_i - r_j}{t} + \frac{\ell_i - \ell_j}{t(\log |t|^2)} \right|^2 \cdot \frac{g_i g_j}{g^2} \\ &\quad + \frac{1}{2} \sum_{i,j} \left(\frac{\partial_{t\bar{t}} \varphi_i}{\varphi_i} + \frac{\partial_{t\bar{t}} \varphi_j}{\varphi_j} \right) \frac{g_i g_j}{g^2} + \frac{1}{2} \sum_{i,j} \Re \left(\frac{\partial_t \varphi_i}{\varphi_i} \cdot \overline{\frac{\partial_t \varphi_j}{\varphi_j}} \right) \frac{g_i g_j}{g^2} \\ &\quad + \sum_{i < j} \Re \left\{ \left(\frac{r_i - r_j}{t} + \frac{\ell_i - \ell_j}{t(\log |t|^2)} \right) \cdot \left(\frac{\partial_t \varphi_i}{\varphi_i} - \frac{\partial_t \varphi_j}{\varphi_j} \right) \right\} \frac{g_i g_j}{g^2}. \end{aligned}$$

(2) If $r_i \geq 0$ and $0 \leq \ell_i \leq N$ for all $i \in I$, then as $t \rightarrow 0$

$$\begin{aligned} \partial_{t\bar{t}} \log g &= -\frac{1}{2} \sum_{i,j} \frac{\ell_i + \ell_j}{|t|^2 (\log |t|^2)^2} \cdot \frac{g_i g_j}{g^2} + \frac{1}{2} \sum_{i,j} \left| \frac{r_i - r_j}{t} + \frac{\ell_i - \ell_j}{t (\log |t|^2)} \right|^2 \cdot \frac{g_i g_j}{g^2} \\ &\quad + O\left(\frac{(-\log |t|)^{2N}}{|t|g(t)^2}\right). \end{aligned}$$

Proof. (1) The first equality of (1) follows from Lemma 3.1 (1). Since

$$\partial_{t\bar{t}} g(t) = \sum_{i \in I} \left(-\frac{\ell_i}{|t|^2 (\log |t|^2)} + \frac{\partial_{t\bar{t}} \varphi_i}{\varphi_i} - \frac{|\partial_t \varphi_i|^2}{\varphi_i^2} + \left| \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right|^2 \right) g_i$$

by Lemma 3.1 (2), we get

$$\begin{aligned} (3.1) \quad g \partial_{t\bar{t}} g &= \sum_{i,j \in I} g_j \left(-\frac{\ell_i}{|t|^2 (\log |t|^2)} + \frac{\partial_{t\bar{t}} \varphi_i}{\varphi_i} - \frac{|\partial_t \varphi_i|^2}{\varphi_i^2} + \left| \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right|^2 \right) g_i \\ &= \frac{1}{2} \sum_{i,j \in I} \left(-\frac{\ell_i + \ell_j}{|t|^2 (\log |t|^2)} + \frac{\partial_{t\bar{t}} \varphi_i}{\varphi_i} + \frac{\partial_{t\bar{t}} \varphi_j}{\varphi_j} - \frac{|\partial_t \varphi_i|^2}{\varphi_i^2} - \frac{|\partial_t \varphi_j|^2}{\varphi_j^2} \right. \\ &\quad \left. + \left| \frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right|^2 + \left| \frac{r_j}{t} + \frac{\ell_j}{t (\log |t|^2)} + \frac{\partial_t \varphi_j}{\varphi_j} \right|^2 \right) g_i g_j. \end{aligned}$$

By the first equality of Lemma 3.2 (1), we get

$$\begin{aligned} (3.2) \quad |\partial_t g|^2 &= \sum_{i,j \in I} \left(\frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right) \overline{\left(\frac{r_j}{t} + \frac{\ell_j}{t (\log |t|^2)} + \frac{\partial_t \varphi_j}{\varphi_j} \right)} g_i g_j \\ &= \sum_{i,j \in I} \Re \left(\frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right) \overline{\left(\frac{r_j}{t} + \frac{\ell_j}{t (\log |t|^2)} + \frac{\partial_t \varphi_j}{\varphi_j} \right)} g_i g_j. \end{aligned}$$

By (3.1) and (3.2), we get

$$\begin{aligned} (3.3) \quad g \partial_{t\bar{t}} g - |\partial_t g|^2 &= \frac{1}{2} \sum_{i,j \in I} \left(-\frac{\ell_i + \ell_j}{|t|^2 (\log |t|^2)} + \frac{\partial_{t\bar{t}} \varphi_i}{\varphi_i} + \frac{\partial_{t\bar{t}} \varphi_j}{\varphi_j} - \frac{|\partial_t \varphi_i|^2}{\varphi_i^2} - \frac{|\partial_t \varphi_j|^2}{\varphi_j^2} \right. \\ &\quad \left. + \left| \left(\frac{r_i}{t} + \frac{\ell_i}{t (\log |t|^2)} + \frac{\partial_t \varphi_i}{\varphi_i} \right) - \left(\frac{r_j}{t} + \frac{\ell_j}{t (\log |t|^2)} + \frac{\partial_t \varphi_j}{\varphi_j} \right) \right|^2 \right) g_i g_j. \end{aligned}$$

Since $\partial_{t\bar{t}} \log g = (g \partial_{t\bar{t}} g - |\partial_t g|^2)/g^2$, the second equality of Lemma 3.2 (1) follows from (3.3). This proves (1).

(2) By the definition of $g_i(t)$, we get

$$(3.4) \quad \left(\frac{\partial_{t\bar{t}} \varphi_i}{\varphi_i} + \frac{\partial_{t\bar{t}} \varphi_j}{\varphi_j} \right) \frac{g_i g_j}{g^2} = \frac{(\varphi_i \partial_{t\bar{t}} \varphi_j + \varphi_j \partial_{t\bar{t}} \varphi_i) \cdot |t|^{2(r_i+r_j)} (\log |t|^2)^{\ell_i+\ell_j}}{g^2},$$

(3.5)

$$\left(\frac{\partial_t \varphi_i}{\varphi_i} - \frac{\partial_t \varphi_j}{\varphi_j} \right) \frac{g_i g_j}{g^2} = \frac{(\varphi_j \partial_t \varphi_i - \varphi_i \partial_t \varphi_j) \cdot |t|^{2(r_i+r_j)} (\log |t|^2)^{\ell_i+\ell_j}}{g^2},$$

(3.6)

$$\left(\frac{\partial_t \varphi_i}{\varphi_i} \cdot \overline{\frac{\partial_t \varphi_j}{\varphi_j}} \right) \frac{g_i g_j}{g^2} = \frac{(\partial_t \varphi_i \cdot \overline{\partial_t \varphi_j}) \cdot |t|^{2(r_i+r_j)} (\log |t|^2)^{\ell_i+\ell_j}}{g^2}.$$

Since the functions $\varphi_i \partial_{t\bar{t}} \varphi_j + \varphi_j \partial_{t\bar{t}} \varphi_i$, $\varphi_j \partial_t \varphi_i - \varphi_i \partial_t \varphi_j$, $\partial_t \varphi_i \cdot \overline{\partial_t \varphi_j}$ are bounded near $t = 0$ and since $|t|^{2(r_i+r_j)} (-\log |t|^2)^{\ell_i+\ell_j} \leq (-\log |t|^2)^{2N}$ by the definition of N , we get (2) by the second equality of Lemma 3.2 (1) and (3.4), (3.5), (3.6). \square

Lemma 3.3. *Let $\varphi_i \in C_{\mathbf{R}}^\infty(T)$ for $0 \leq i \leq N$ and set $g(t) = \sum_{i=0}^N (\log |t|^2)^i \varphi_i(t)$. Assume that $g(t) > 0$ on T^o and that $\varphi_i(0) \neq 0$ for some $0 \leq i \leq N$. Set*

$$\ell := \max_{0 \leq i \leq N, \varphi_i(0) \neq 0} \{i\} \in \mathbf{Z}_{\geq 0}.$$

Then there exists a constant $C > 0$ such that the following inequalities hold

$$|\partial_t \log g(t)| \leq \frac{C}{|t|(-\log |t|)}, \quad \left| \partial_{t\bar{t}} \log g(t) + \frac{\ell}{|t|^2(-\log |t|)^2} \right| \leq \frac{C}{|t|^2(-\log |t|)^3}.$$

Proof. Set $I = \{0, 1, \dots, N\}$ and $g_i(t) := (-\log |t|)^i \varphi_i(t)$ for $i \in I$. Namely, we set $(r_i, \ell_i) = (0, i)$ in Lemma 3.2. Since $g(t) = \varphi_\ell(0)(-\log |t|)^\ell (1 + O(1/\log |t|))$ as $t \rightarrow 0$, we get for each $0 \leq i \leq N$ the following asymptotic behavior as $t \rightarrow 0$:

$$(3.7) \quad \left| \frac{g_i(t)}{g(t)} \right| = \begin{cases} O(|t|(-\log |t|)^i) & (i > \ell), \\ 1 + O(|t|(\log |t|)^n) & (i = \ell), \\ O((-\log |t|)^{-(\ell-i)}) & (i < \ell). \end{cases}$$

By the first equality of Lemma 3.2 (1) and (3.7), there are constants $C, C' > 0$ such that

$$|\partial_t \log g(t)| \leq \frac{C}{|t|(-\log |t|)} \sum_{i=0}^N \left| \frac{g_i}{g} \right| \leq \frac{C'}{|t|(-\log |t|)}.$$

This proves the first inequality. Since $g(t) = \varphi_\ell(0)(-\log |t|)^\ell (1 + O(1/\log |t|))$, there exists $c > 0$ such that $g(t) \geq c > 0$ on T^o . In particular $O(1/g(t)) = O(1)$. This, together with Lemma 3.2 (2), yields that

(3.8)

$$\begin{aligned} \partial_{t\bar{t}} \log g &= -\frac{\ell}{|t|^2(\log |t|^2)^2} - \frac{1}{2} \sum_{(i,j) \neq (\ell,\ell)} \frac{i+j}{|t|^2(\log |t|^2)^2} \left(\frac{g_i}{g} \right) \left(\frac{g_j}{g} \right) \\ &\quad + \frac{1}{2} \sum_{i \neq j} \frac{(i-j)^2}{|t|^2(\log |t|^2)^2} \left(\frac{g_i}{g} \right) \left(\frac{g_j}{g} \right) + O\left(\frac{(-\log |t|)^{2N}}{|t|} \right). \end{aligned}$$

Since $|g_i(t)g_j(t)/g(t)^2| = O(1/\log |t|)$ when $i \neq j$ by (3.7), the second and the third term in the right hand side of (3.8) is bounded by $|t|^{-2}(-\log |t|)^{-3}$ as $t \rightarrow 0$. Similarly, it follows from (3.7) that the second term of the right hand side of (3.8) is bounded by $|t|^{-2}(-\log |t|)^{-3}$. The second inequality follows from (3.8). \square

4. THE BOUNDARY BEHAVIOR OF THE CURVATURE OF THE L^2 -METRIC

In this section, we define $N, \ell_q \in \mathbf{Z}_{\geq 0}$ as

$$N := n\rho_q, \quad \ell_q := \max_{0 \leq i \leq N, a_i(0) \neq 0} \{i\},$$

where $a_i(t) \in C^\infty(T)$, $0 \leq i \leq N$, are the functions in Theorem 2.1. Recall that the integer $\nu > 0$ was defined in Sect. 2.

4.1. The singularity of the first Chern form.

Theorem 4.1. *The following formula holds as $s \rightarrow 0$:*

$$c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) = \left\{ \frac{\ell_q}{|s|^2 (\log |s|)^2} + O\left(\frac{1}{|s|^2 (\log |s|)^3}\right) \right\} \sqrt{-1} ds \wedge d\bar{s}$$

Proof. Recall that T is another unit disc and that the map $\mu: T \rightarrow S$ is defined as $s = \mu(t) = t^\nu$. By Theorem 2.1, we get

$$\begin{aligned} \mu^* c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) &= -\frac{\sqrt{-1}}{2\pi} \mu^* \partial \bar{\partial} \log \det G(s) = -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \det H(t) \\ &= -\frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log \left[\sum_{m=0}^N a_m(t) (\log |t|^2)^m \right]. \end{aligned}$$

We set $g(t) = \det H(t) = \sum_{i=0}^N (\log |t|^2)^i a_i(t)$ in Lemma 3.3. Since $a_i(0) \neq 0$ for some $0 \leq i \leq N$ by Theorem 2.1, we deduce from Lemma 3.3 that

(4.1)

$$\mu^* c_1(R^q \pi_* \omega_{X/S}(\xi), h_{L^2}) = \ell_q \frac{\sqrt{-1} dt \wedge d\bar{t}}{|t|^2 (-\log |t|)^2} + O\left(\frac{\sqrt{-1} dt \wedge d\bar{t}}{|t|^2 (-\log |t|)^3}\right).$$

Since $\mu^* \{\sqrt{-1} ds \wedge d\bar{s} / (|s|^2 (-\log |s|)^m)\} = \nu^{2-m} \sqrt{-1} dt \wedge d\bar{t} / (|t|^2 (-\log |t|)^m)$, the desired inequality follows from (4.1). \square

Remark 4.2. The Hermitian metric $\mu^* \det h_{L^2}$ on the line bundle $\det R^q f_* \omega_{Y/T}(\xi)$ is good in the sense of Mumford [9]. Namely, the following estimates hold:

(1) There exist constants $C, \ell > 0$ such that

$$\det H(t) \leq C(-\log |t|^2)^\ell, \quad (\det H(t))^{-1} \leq C(-\log |t|^2)^\ell.$$

(2) There exists a constant $C > 0$ such that

$$|\partial_t \log \det H(t)| \leq \frac{C}{|t|(-\log |t|)}, \quad |\partial_{\bar{t}} \log \det H(t)| \leq \frac{C}{|t|^2(-\log |t|)^2}.$$

The inequalities (1) follow from Theorem 2.1. By setting $g(t) = \det H(t)$ in Lemma 3.3, we get (2) because $\det H(t) = g(t) = \sum_{i=0}^N (\log |t|^2)^i a_i(t)$, $a_i(t) \in C_{\mathbf{R}}^\infty(T)$ with $a_i(0) \neq 0$ for some $0 \leq i \leq N$ by Theorem 2.1.

We do not know if the L^2 -metric $\mu^* h_{L^2}$ on $R^q f_* \omega_{Y/T}(F^* \xi)$ is good in the sense of Mumford, because the estimates

$$\|\partial_t H \cdot H^{-1}\| \leq \frac{C}{|t|(-\log |t|)}, \quad \|\partial_{\bar{t}}(\partial_t H \cdot H^{-1})\| \leq \frac{C}{|t|^2(-\log |t|)^2}$$

do not necessarily follow from Theorem 2.1; from Theorem 2.1, we have only the estimates $\|\partial_t H \cdot H^{-1}\| \leq C(-\log |t|)^\ell / |t|$ and $\|\partial_{\bar{t}}(\partial_t H \cdot H^{-1})\| \leq C(-\log |t|)^\ell / |t|^2$, where $\|A\| = \sum_{i,j} |a_{ij}|$ for a matrix $A = (a_{ij})$.

4.2. Proof of Theorem 1.1. Let $\lambda_1, \dots, \lambda_{\rho_q}$ be the eigenvalues of the Hermitian endomorphism $\sqrt{-1}\mathcal{R}(s)$. By the Nakano semi-positivity of $(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$, we get $\lambda_\alpha \geq 0$ for all $1 \leq \alpha \leq \rho_q$. By Theorem 4.1, we have the following inequality on S^o

$$0 \leq \sqrt{-1} \operatorname{Tr}[\mathcal{R}(s)] = \sum_{\alpha} \lambda_{\alpha} \leq \frac{C}{|s|^2(-\log|s|)^2}.$$

In particular, we get $\Lambda := \max_{\alpha} \{\lambda_{\alpha}\} \leq C/(|s|^2(-\log|s|)^2)$. We get the desired inequality for $\sqrt{-1}\mathcal{R}(s)$ from the inequality $\sqrt{-1}\mathcal{R}(s) \leq \Lambda \cdot \operatorname{Id}_{R^q\pi_*\omega_{X/S}(\xi)}$. The inequality for $c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$ is already proved in Theorem 4.1. This completes the proof. \square

4.3. Proof of Theorem 1.2. By the curvature formula for Quillen metrics [3], the following equation of currents on S^o holds

$$(4.2) \quad \begin{aligned} & -dd^c \log \tau(X/S, \omega_{X/S}) + \sum_{q \geq 0} (-1)^q c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2}) \\ & = [\pi_*\{\operatorname{Td}(TX/S, h_{X/S}) \operatorname{ch}(\omega_{X/S}(\xi))\}]^{(2)}, \end{aligned}$$

where $[A]^{(p)}$ denotes the component of degree p of a differential form A . By [13, Lemma 9.2], there exists $r \in \mathbf{Q}_{>0}$ such that as $s \rightarrow 0$

$$(4.3) \quad [\pi_*\{\operatorname{Td}(TX/S, h_{X/S}) \operatorname{ch}(\omega_{X/S}(\xi))\}]^{(2)}(s) = O\left(\frac{\sqrt{-1}|s|^{2r}(-\log|s|)^n ds \wedge d\bar{s}}{|s|^2}\right).$$

By Theorem 1.1, we get

$$(4.4) \quad \sum_{q \geq 0} (-1)^q c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2}) = \frac{\sum_{q \geq 0} (-1)^q \ell_q}{2\pi} \frac{\sqrt{-1} ds \wedge d\bar{s}}{|s|^2(-\log|s|)^2} + O\left(\frac{\sqrt{-1} ds \wedge d\bar{s}}{|s|^2(-\log|s|)^3}\right).$$

By (4.2), (4.3), (4.4), we get on S^o

$$\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \tau(X/S, \omega_{X/S}) = \frac{\sum_{q \geq 0} (-1)^q \ell_q}{2\pi} \frac{\sqrt{-1} ds \wedge d\bar{s}}{|s|^2(-\log|s|)^2} + O\left(\frac{\sqrt{-1} ds \wedge d\bar{s}}{|s|^2(-\log|s|)^3}\right).$$

This completes the proof. \square

5. CANONICAL SINGULARITIES AND THE CURVATURE OF L^2 -METRIC

In this section, we assume that the central fiber X_0 is reduced and irreducible and has only canonical (equivalently rational) singularities. Then $G(s) = (G_{\alpha\bar{\beta}}(s))$ is expected to have better regularity than usual. To see this, set

$$\mathcal{B}(S) := C^\infty(S) \oplus \bigoplus_{r \in \mathbf{Q} \cap (0,1]} \bigoplus_{k=0}^n |s|^{2r} (\log|s|)^k C^\infty(S) \subset C^0(S).$$

By [14, Th. 7.2], the L^2 -metric h_{L^2} on $R^q\pi_*\omega_{X/S}(\xi)$ is a continuous Hermitian metric lying in the class $\mathcal{B}(S)$. Namely, $G_{\alpha\bar{\beta}}(s) \in \mathcal{B}(S)$ for all $1 \leq \alpha, \beta \leq \rho_q$.

Proposition 5.1. *If X_0 has only canonical singularities, then there exists $r \in \mathbf{Q}_{>0}$ and $C > 0$ such that the following inequality of real $(1,1)$ -forms on S^o holds*

$$0 \leq c_1(R^q\pi_*\omega_{X/S}(\xi), h_{L^2}) \leq C \frac{\sqrt{-1}|s|^{2r} ds \wedge d\bar{s}}{|s|^2(-\log|s|)^2}.$$

In particular, the curvature $i\mathcal{R}(s)ds \wedge d\bar{s}$ satisfies the following estimate:

$$0 \leq \sqrt{-1}\mathcal{R}(s) \leq \frac{C|s|^{2r}}{|s|^2(-\log|s|)^2} \text{Id}_{R^q\pi_*\omega_{X/S}(\xi)}.$$

Proof. Since $G_{\alpha\bar{\beta}}(s)$ is continuous on S , we may assume by an appropriate choice of basis that $G_{\alpha\bar{\beta}}(0) = \delta_{\alpha\beta}$. Since $\det G(s) \in \mathcal{B}(S)$, there exist a finite set I and $(r_i, \ell_i) \in \mathbf{Q}_{>0} \times \mathbf{Z}_{\geq 0}$ for each $i \in I$ such that

$$\det G(s) = 1 + \sum_{i \in I} |s|^{2r_i} (\log|s|^2)^{\ell_i} \varphi_i(s).$$

We set $r_0 = 0$, $\ell_0 = 0$ and $\varphi_0(s) = 1$. For $i \in \{0\} \cup I$, we set $g_i(s) := |s|^{2r_i} (\log|s|^2)^{\ell_i} \varphi_i(s)$. By Lemma 3.2 (2) applied to $\det G(s)$, we get

$$\begin{aligned} & -\partial_{s\bar{s}} \log \det G(s) \\ &= \frac{1}{2} \sum_{i,j \in I \cup \{0\}} \frac{\ell_i + \ell_j}{|s|^2 (\log|s|^2)^2} \cdot \frac{g_i g_j}{g^2} - \frac{1}{2} \sum_{i,j \in I \cup \{0\}} \left| \frac{r_i - r_j}{s} + \frac{\ell_i - \ell_j}{s(\log|s|^2)} \right|^2 \frac{g_i g_j}{g^2} \\ & \quad + O\left(\frac{(-\log|t|)^{2N}}{|t|}\right) \\ &\leq C \sum_{i,j \in I \cup \{0\}} \frac{(\ell_i + \ell_j)|s|^{2(r_i+r_j)}}{|s|^2 (\log|s|^2)^2} + C \sum_{i,j \in I \cup \{0\}} \left| \frac{r_i - r_j}{s} + \frac{\ell_i - \ell_j}{s(\log|s|^2)} \right|^2 |s|^{2(r_i+r_j)} \\ & \quad + O\left(\frac{(-\log|t|)^{2N}}{|t|}\right). \end{aligned}$$

Set $r := \min_{i \in I} \{r_i\} > 0$. Since $r_i + r_j > r$ for all $(i, j) \in (I \cup \{0\}) \times (I \cup \{0\}) \setminus \{(0, 0)\}$, we get

$$-\partial_{s\bar{s}} \log \det G(s) \leq C \frac{2\ell_0}{|s|^2 (\log|s|^2)^2} + C \sum_{(i,j) \neq (0,0)} \frac{|s|^{2(r_i+r_j)}}{|s|^2 (\log|s|^2)^2} \leq \frac{C|s|^{2r}}{|s|^2 (\log|s|^2)^2}.$$

because $\ell_0 = 0$. Since $-\partial_{s\bar{s}} \log \det G(s) \geq 0$ by the Nakano semi-positivity of $(R^q\pi_*\omega_{X/S}(\xi), h_{L^2})$ by [2], [7], we get the first inequality. The proof of the second inequality is the same as that of the corresponding inequality of Theorem 1.1. \square

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